4.1 (Long run behavior of MC)

$$P^n \to \begin{bmatrix} \frac{b}{a+b} & \frac{b}{a+b} \\ \frac{b}{a+b} & \frac{b}{a+b} \end{bmatrix} = \begin{bmatrix} \pi \\ \pi \end{bmatrix}$$

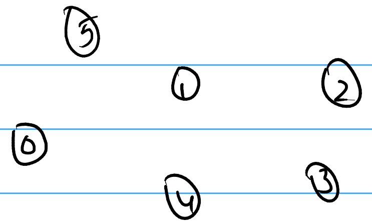
$\pi = \pi P$ (Eigenvalue equation for $\lambda = 1$.)

Turns out that this always happens if P is a regular transition matrix

This phenomenon is referred to as follows:
"The MC forgets its starting point in the long run"

Regular transition prob. $S = \{0, 1, \dots, N\}$
P is called regular is $\exists k$ st
P^k has all positive entries.

$$(P^k)_{ij} > 0 \quad \forall \begin{matrix} i = 0, \dots, N \\ j = 0, \dots, N \end{matrix}$$



Suppose I know this chain is regular with $k = 3$.

$$P_{ij}^{k} = \underbrace{P(X_k = j \mid X_0 = i)}_{} > 0$$

prob. of going from i to j in k steps.

$\Rightarrow$ I can go from any i to any j in k steps with +ve prob.

S forms 1 SINGLE COMMUNICATING CLASS.

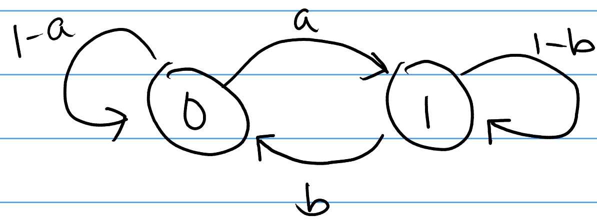
Thm: If P is regular then $\exists$ a unique stationary prob. π

1) $\sum_{i=0}^{N} \pi_i = 1$

2) $\pi = \pi P$

3) $\lim_{n \to \infty} P^n = \begin{bmatrix} \pi \\ \vdots \\ \pi \end{bmatrix}_{(N+1) \times (N+1)}$

Two step MC : $P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$



$a = 0 \quad b = 0$ (we cannot have this)
if we want regularity
$a = 1 \quad b = 1$

$P^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad P^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\Rightarrow P^3 = P^2 P \, , \; P^4 = I$

$\lim P^n$ = Does not exist.

P is not regular.

P will turn out to have PERIOD 2.

communication classes.

Regular is also called "Irreducible, Aperiodic".

2 step MC is regular if $0 < a, b, < 1$.

★ lovely example. Social classes of successive generations can be regarded as a Markov Chain.

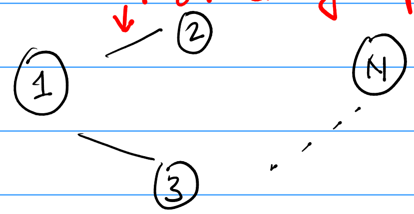
★ Could do a jupyter notebook.

$$
\begin{array}{cc|ccc}
 & & & \text{Son} & \\
 & & L & M & U \\
 & L & 0.4 & 0.5 & 0.10 \\
\text{Father} & M & 0.05 & 0.7 & 0.25 \\
 & U & 0.05 & 0.5 & 0.45
\end{array}
$$

In the long term, what fraction of the people are in the middle class in the long term?

Lets return to regular matrices:

indicating possible transitions.



N states.

Then $\exists$ m st $P^m_{ij} > 0 \quad \forall i,j$

If P is regular we cannot get stuck in any state. So if we start in state i there is at least one other state that was not previously reached that was reachable. So we may form a path $i \ i_1 \ i_2 \cdots i_k \cdots j$ where each i_k is some new state that was not previously reached. (This is because if any state is repeated then there is a loop in the path that we can erase)

Thus there is a path of AT MOST N steps from i to j and let the length of this path be $m(i,j)$. We have shown that

$$1 \leq m(i,j) \leq N$$

prob of going from i to j in $m(i,j)$ steps

Thus $$P_{ij}^{m(i,j)} = \mathbb{P}\big(\overbrace{X_{m(i,j)} = j \mid X_0 = i}^{A}\big)$$

$$\geq \mathbb{P}\big(\underbrace{\text{we take a path } \gamma \text{ from } i \text{ to } j}_{B}\big) > 0$$

This is because $B \subseteq A$ (B implies that A happens)

Will prove that $\boxed{P_{ij}^{N^2}} > 0 \quad \forall ij$ and further $P_{ij}^k > 0 \quad \forall\, k > N^2$. (Eventually large enough of powers of P will have all +ve entries)

We will prove this when we do periodicity

Next. Suppose $P^n \to \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$ that is

$P_{kj}^n \to \pi_j \quad \forall\, kj$ (note this is indep of the row k)

Theorem: $\pi = (\pi_1, \dots \pi_N)$ solves

1) $\pi = \pi P$ the eigenvalue equation

2) π is UNIQUE

linear algebra

$$(A.B)_{ij} = \sum A_{ik} B_{kj}$$

Easy to prove 1.

$$P^n = P^{n-1} P$$

$$P^n_{kj} = \sum_{\ell=1}^{N} P^{n-1}_{k\ell} P_{\ell j}$$

$$\downarrow$$

$$\pi_j = \sum_{\ell=1}^{N} \pi_\ell P_{\ell j}$$

(π is a row vector)

$$\pi = \pi P.$$

2) let $x = (x_1 \cdots x_N)$ be a probability vector solving

$$x = xP \iff x_j = \sum_i x_i P_{ij} = (\ast 1)$$

Note $xP = xP^2$ from above $\Rightarrow xP^2 = xP^3 \cdots = xP^{n-1}$

$$\lim_{n\to\infty} x_j = \lim_{n\to\infty} \sum_i x_i P_{ij}^{(n)} = \sum_{i=1}^{N} x_i \pi_j = \pi_j$$

row sums to 1

Ex:
$$\begin{bmatrix} 0.4 & 0.5 & 0.10 \\ 0.05 & 0.7 & 0.25 \\ 0.05 & 0.5 & 0.45 \end{bmatrix}$$
(Social class matrix)

What is π? $\left(\frac{1}{13}, \frac{5}{8}, \frac{31}{104}\right)$

4.1.1 Doubly stochastic Matrices.

Let P be regular.

We know $\sum_{j=1}^{N} P_{ij} = 1$ (if you start in state i you must go to SOME other state j in the next step). [STOCHASTIC MATRIX]

If $\sum_{i=1}^{N} P_{ij} = 1$ (Column sum) $\forall i$

then P is called Doubly Stochastic.

Let $\pi = \left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$

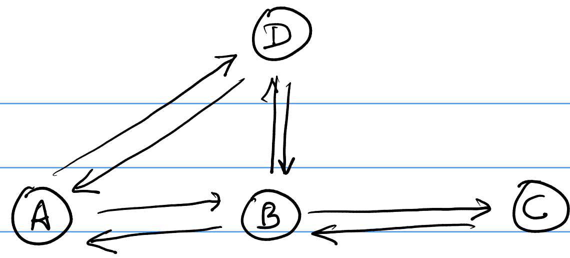
doubly stochastic

$$(\pi P)_j = \sum_{k=1}^{N} \pi_k P_{kj} = \sum_{k=1}^{N} \frac{1}{N} P_{kj} = \frac{1}{N}$$
$$= \pi_j$$

True for every j. By uniqueness, this is the UNIQUE stationary probability

If you have doubly stochastic matrices, its equally likely to be in each state in the long term.

Ex:



Railroad lines connecting 4 towns. Train randomly chooses a town it is connected to and traverses it (uniform probability)

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \end{array}$$

What is π? Note that the outdegree of the 4 vertices is $(2, 3, 1, 2)$

Using $\pi P = \pi$

$$\pi_j = \pi_k P_{kj}$$

Try $\pi = (2, 3, 1, 2)$ $\quad 2 = 2 \cdot 0 + 3 \cdot \frac{1}{3} + 1 \cdot 0 + 2 \cdot \frac{1}{2}$

$= 2$ ✓

$3 = 2 \cdot \frac{1}{2} + 1 \cdot 1 + 2 \cdot \frac{1}{2} = 3$ ✓

and so on. So certainly

$\pi P = \pi$

What must we do to turn π into a probability? $\pi = (2, 3, 1, 2)$ does not sum to 1. $\left(\frac{2}{8}, \frac{3}{8}, \frac{1}{8}, \frac{2}{8}\right)$

We know that π must be unique. We seem to have 2 vectors that solve $\pi P = P$. What is going on?

4.1.2 Interpretation of Limiting distribution

$$\pi_j = \lim_{n\to\infty} P_{ij}^{(n)} = \lim_{n\to\infty} P\{X_n = j \mid \underline{X_0 = i}\}$$

"Probability of finding the chain in state j given that it has been running for a long time and it started in state i".

☆ Quiz or HW. Suppose I start a chain on N states with $P(X_0=1)=\pi_1, \ldots, P(X_0=N)=\pi_N$. What is $P(X_k=1), \ldots P(X_k=N)$?

2nd interpretation: π_j also gives long run mean fraction of time spent in state j.

How to describe this?

$$\underset{\text{mean}}{E}\Big[\underbrace{\frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i=j\}}}_{\text{fraction of time in state } j}\Big|\underbrace{X_0=k}_{\text{initial state.}}\Big] = \frac{1}{n}\sum_{i=1}^{n} P(X_i=j \mid X_0=k)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \overbrace{P(X_i=j \mid X_0=k)}^{\text{conditional prob.}} = \underbrace{\frac{1}{n}\sum_{i=1}^{n} P_{kj}^{i}}_{\to \pi_j} \quad ☆1$$

We know $P_{kj}^{i} \longrightarrow \pi_j$ (by assumption)

Claim: if $\{a_i\}_{i=1}^{\infty}$ $a_n \to a$ then

$\frac{1}{n}\sum_{i=1}^{n} a_i \xrightarrow{n\to\infty} a$ as well] $\leftarrow$ R. Calc 2. stuff.

((called convergence of the Cesaro mean)
There are a bunch of cool related theorems: see Abel, Tauberian theorem, etc)

Good problems: 4.1.9 and 4.1.10

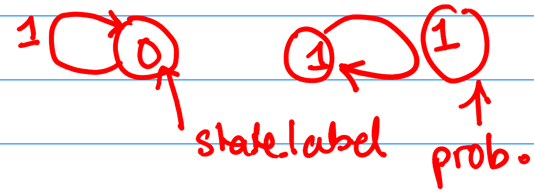
Section 4.2 has a bunch of cool examples.

- "Markov chains" dependent on k previous states
- Reliability and Redundancy
- Age replacement policies
- Optimal replacements

(Recommend reading section 4.2 from the textbook).

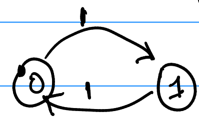
4.3 The classification of states.

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$ is not regular. $\{0\}$ and $\{1\}$ are



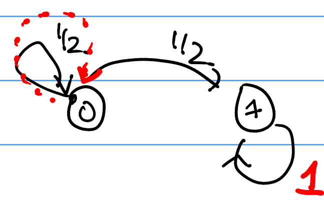
both absorbing states

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$ is also not regular.



(we saw this in the previous lecture)

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1/2 & 1/2 \\ 1 & 0 & 1 \end{array}$$, then



is also not regular

$$P^n = \begin{array}{cc} 0 & 1 \\ \left[\begin{matrix} (1/2)^n & 1-(1/2)^n \\ \boxed{0}_{21} & 1 \end{matrix}\right] \end{array}$$ ← geometric prob of staying in 0.

$\Rightarrow \lim_{n\to\infty} P^n = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ (again not regular)

Since $P^n_{00} \to 0 \Rightarrow 0$ is called a transient state. 1 is absorbing (recurring state)

P is regular $\Leftrightarrow$ P irreducible and aperiodic

Irreducible chains:

Communication: i, j communicate if $P^{(n)}_{ij} > 0$, $P^{(m)}_{ji} > 0$ for some $m, n \geq 0$. $\boxed{i \leftrightarrow j}$ if i and j communicate.

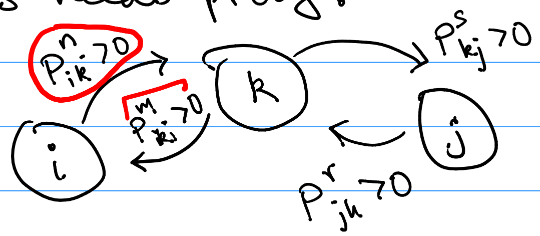
1. Reflexive: $i \leftrightarrow i$ (communicates with itself)

   since $P^{(0)}_{ii} = 1$ $\quad (\mathrm{Prob}(X_0 = i \mid X_0 = i) = 1)$

2. Symmetry $i \leftrightarrow j \Leftrightarrow j \leftrightarrow i$ (by definition)

3. Transitivity $i \leftrightarrow k$ and $k \leftrightarrow j$ then $i \leftrightarrow j$

This needs proof.



keep just $t = k$ term.

$$\Rightarrow \underbrace{P^{n+s}_{ij} = \sum_{t=1}^{N} P^n_{it} P^s_{tj}} \geq P^n_{ik} P^s_{kj} > 0$$

Called Chapman-Kolmogorov equation.

similarly $P^{(m+r)}_{ji} > 0$

So take states $S = \{\underbrace{1, 2}_{\text{equivalence class 1}} \cdots \overbrace{\cdots}^{\text{equiv class 2}}, \underbrace{N}_{\text{equiv class 3}}\}$

You can break up states into equivalence classes.

Irreducible: An MC is irred if it has ONLY one communication (or equivalence) class.

$$\text{Ex } P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1/2 & 1/2 & 0 & 0 \\ 2 & 1/4 & 3/4 & 0 & 0 \\ 3 & 0 & 0 & 1/3 & 2/3 \\ 4 & 0 & 0 & 2/3 & 1/3 \end{array}$$

$\{1,2\}$ $P^1_{12} > 0$, $P^1_{21} > 0$

$\{3,4\}$

Q: How many equiv classes? 2

$$P = \begin{array}{c|cccccc} & 1 & 2 & & & N-1 & N \\ \hline 1 & 1 & 0 & \cdots & & & 0 \\ & q & 0 & p & 0 & & \\ & 0 & q & 0 & p & 0 & \cdots \\ & & & & & & \\ N & 0 & \cdots & & & \cdots & 1 \end{array}$$

$q = 1-p$

$\{1\}$, $\{N\}$

$\{2,3,\ldots,N-1\}$

Random Walk.

Q: What are the comm. classes?

Recall: I had said "Regular" P

$\iff$ Irreducible + aperiodic MC.

4.3.2

Periodicity

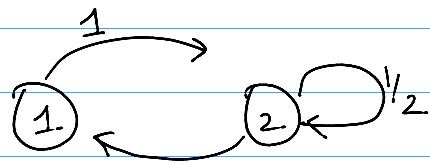
greatest common denom.

$$d(i) = \gcd\{n \geq 1 \mid \underbrace{P^n(i,i) > 0}_{\text{non zero prob of return to } i \text{ in } n \text{ steps.}}\}$$

$d(i)$ = PERIOD of i

If $P^{(n)}_{ii} = 0 \ \forall\ n \geq 1$ define $\underbrace{d(i) = 0}_{\star \text{ HW or QUIZ.}}$

Ex:



$\frac{1}{2} = P^2_{11} > 0$ $\qquad$ $1 = P^3_{11} > 0$

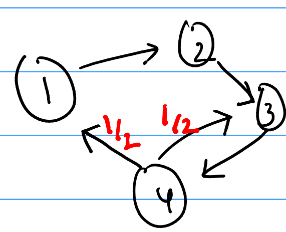
$d(1) = \gcd\{2, \underline{3}\ \underline{4}\ \underline{5} \ldots\}$
$= 1$

$d(2) = \gcd\{1, 2, 3, \ldots\} = 1$?

Notice that $d(1) = d(2) = 1$! (A periodic)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array}$$

Fi periodicities.



$$d(1) = \gcd\{\overset{P_{11}^4 = \frac{1}{2}}{4}, 6, 8, \dots\} = 2$$

$$d(2) = \gcd\{4, 6, 8, \dots\} = 2$$

and similarly $d(3) = d(4) = 2$

HW: Ex from page 24 in KP (Wet and Dry days)

Thm:

1) If $i \leftrightarrow j$ Then $d(i) = d(j)$

2) $\exists\, N$ st $\forall\, n \geq N$

$P_{ii}^{nd(i)} > 0$ (eventually all multiples of $d(i)$ belong to the set)

(for previous ex $N = 4$)

if I can get from j to i in m steps then $P_{ji}^{m+nd(i)} > 0$ for all large enough n

APERIODIC: All states have $d(i) = 1$.

Euclidean algorithm, Bezouts identity etc.

Theorem: let $J \subseteq \mathbb{Z}^+$ be closed under addition.

That is $m, n \in J \Rightarrow m+n \in J$.

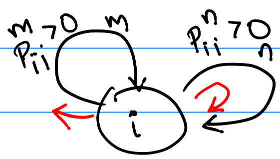
let $d = \gcd(J)$

Then 1) $J \subset \{0, d, 2d, \dots\}$

2) $\exists N$ st $nd \in J \quad \forall n \geq N$

(or $\{0, d, 2d, \dots\} \setminus J$ is finite)

Notice that $Q = \{ n \geq 1 \mid P^n_{ii} > 0 \text{ is such a set} \}$



if $m, n \in Q$

$\Rightarrow m+n \in Q \quad (P^{n+m}_{ii} > 0)$

, (To skip)] How to prove this thing using number Theory.

Lemma: 1) If m, n relatively prime

$(d(m,n) = 1) \Rightarrow \{xm + ny : x, y \in \text{pos. integers}\}$

contains all but a finite # of numbers.

2) let $J \subset \mathbb{Z}^+$ $d = \gcd(J) > 1$, J closed

under addition $\{0, d, 2d, \dots\} \setminus J$ is finite.

Proof of the lemma required for periodicity

These two things are fairly easy to obtain once you have Bezout's identity: given a, b $\exists$ $x, y \in \mathbb{Z}$ st $ax + by = d$ where $d = \gcd(a,b)$

In fact any $az + bt$ is divisible by d.

Pf: let $S = \{ ax+by : ax+by > 0, x, y \in \mathbb{Z}\}$

$d = \inf S > 0$ and an integer. Let $d = ax+by$

Claim: d divides $az+bt$ $\forall z, t \in \mathbb{Z}$

Pf wlog $az+bt \in S$

$az+bt = kd + r$ $\quad 0 \leq r < d$

$az+bt = k(ax+by) + r$

$\Rightarrow$ $r \in S$ or $r = 0$ but r cannot be in S

$\Rightarrow$ $r = 0$!

In particular $d \mid a$, and $d \mid b$.

Why is d the gcd? Suppose c is another divisor.

Then $a = kc$ $\quad b = jc$ Then $d \mid kc + jc$

$\Rightarrow d = r(k+j)c$ This shows d is a multiple of c.

For $a, b \in \mathbb{Z}^+$

Let $S = \{ax+by : x, y \in \mathbb{Z}^+\}$. Let $d = \gcd(S)$

Claim: $\{0, d, 2d, \ldots\} \setminus S$ is finite.

Pf: Let, $a, b \in S$ and $ax+by = d$ (by Bezout)

Then wlog we may write $ax - by = d$ where $a, b, x, y > 0$.

In particular $ax+by = kd$

$ax+by+ax-by = 2ax = (k+1)d$.

Since S is closed under addition

$kd, (k+1)d \in S$

$2kd, (2k+1)d \in S$

$\vdots$

$\ell kd, (\ell k+1)d \ldots (\ell k+\ell)d \in S$

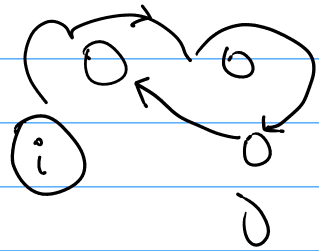
$k^2d, (k^2+1)d, \ldots (k^2+k)d \in S$

Let $j > k^2 \Rightarrow j = nk+\ell$ where $n > k$, $0 \leq \ell < k$

$\Rightarrow jd = \underbrace{(\ell k+\ell)d}_{\in S} + \underbrace{(n-\ell)kd}_{\in S}$

How do you call a state recurrent.

4.3.3 Recurrence and Transience



How to define recurrence?

We could say a state i is recurrent ("something recurs") if $X_n = i$ again and again given $X_0 = i$

OR we could say that $X_n = i$ for some $n \geq 1$

Probabilistically we say i is recurrent if WE ARE SURE that $X_n = i$ for some n. given $X_0 = i$

$$\boxed{i \text{ is recurrent if } \quad P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = 1}$$

* Why not

$\to P^n_{ji} \to 0 \ \forall j$ (transient state)

$$P(X_n = i \text{ for some } n > 1 \mid X_0 = i) = 0.5$$

or some other number?

Consequences of recurrence defn.

We will do this using <u>first returns</u>

$$f_{ii}^{(n)} = P(\text{First return to } i \text{ happens after } n \text{ steps} \mid X_0 = i)$$

$$= P(X_n = i, X_{n-1} \neq i, \ldots X_1 \neq i \mid X_0 = i)$$

$$f_{ii}^{(1)} = P_{ii} \quad (\text{return in 1 step})$$

$$P_{ii}^{(n)} = P(X_n = i \mid X_0 = i)$$

$$= \sum_{k=1}^{n} P(\overbrace{X_n = i}^{A_1}, \underbrace{\text{1st return to } i \text{ on } k^{th} \text{ step}}_{A_2} \mid \underbrace{X_0 = i}_{A_3})$$

$$\rightarrow \left[P(A_1 A_2 \mid A_3) = P(A_1 \mid A_2 A_3) \, P(A_2 \mid A_3) \right]$$

Markov prop.

$$= \sum_{k} P(X_n = i \mid \text{1st return to } i \text{ } \cancel{\text{on}} \overset{X_k = i}{k^{th}} \text{ step}, \cancel{X_0 = i}) \; P(\text{1st return to } i \text{ on } k^{th} \text{ step} \mid X_0 = i)$$

$$= \sum_{k=1}^{n} P(X_n = i \mid X_k = i) \, f_{ii}^{(k)}$$

$$P_{ii}^{n} = \sum_{k=1}^{n} P_{ii}^{n-k} f_{ii}^{(k)}$$

Sometimes, people set $f_{ii}^{(0)} = 0$ and write

$$P_{ii}^{n} = \sum_{k=0}^{n} P_{ii}^{n-k} f_{ii}^{(0)}$$

$$\text{Recurrent} \Rightarrow \overbrace{P(X_n = i \text{ for some } i \mid X_0 = i)}^{f_{ii}} = 1$$

$$P(\text{Ever returning to state } i \mid X_0 = i)$$

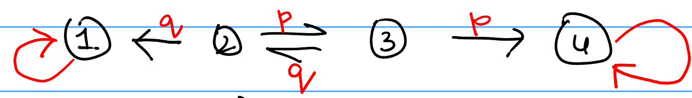
$$= \sum_{k=1}^{\infty} P(\text{1st return in } k \text{ steps} \mid X_0 = i)$$

$$f_{ii} = \sum_{k=0}^{\infty} f_{ii}^{(k)} \qquad \text{after setting } f_{ii}^{(0)} = 0$$

Then i is RECURRENT if

$$\boxed{\sum_{k=0}^{\infty} f_{ii}^{(k)} = 1.}$$

If a state is NOT recurrent we call it Transient.

Ex: 

Comm. classes: $\{1\}, \{4\}, \{2,3\}$

$\{1\}$: recurrent, $\{4\}$: recurrent, $\{2,3\}$: ?

$f_{11}^{1} = 1 \quad f_{11}^{2} = 0 \quad f_{11}^{3} = 0 \ldots \Rightarrow f_{11} = f_{11}^{1} + f_{11}^{2} \cdots = 1$

★ HW or quiz: Show states 2 and 3 are TRANSIENT! ($f_{ii} < 1$)

$$P_{ij}^{n} = \sum_{k=0}^{n} P_{ij}^{n-k} f_{ii}^{(k)}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is $f_{22}^{(k)}$ for $k = 1, 3, \ldots$

$= P(\text{1st return to 2 in 1 step} \mid X_0 = i) = 0 \quad k \text{ odd}$

$f_{22}^{(k)} = (pq)^{k}$ for $k = 2, 3, 4 \ldots$

$$f_{22} = \sum_{k=1}^{\infty} f_{22}^{(k)} = (pq) \frac{1}{1-pq} < 1$$

Multiple Recurrence

Suppose a state is transient $f_{ii} < 1$.

$P(\text{Return to } i \text{ at least twice} \mid X_0 = i)$

$= \sum_{k<j} P(\text{First Return to } i \text{ on } k^{th}, \text{2nd return to } i \text{ on } j^{th} \text{ steps} \mid X_0 = i)$

$= \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} P(\text{2nd Return to } i \text{ on } j^{th} \text{ step} \mid X_k = i)\, P(\text{1st return to } i \text{ on } k^{th} \text{ step} \mid X_0 = i)$

$$= \sum_{k=1}^{\infty} \underbrace{\sum_{j=k+1}^{\infty} f_{ii}^{(j-k)}}_{f_{ii}} f_{ii}^{(k)} = \sum_{k=1}^{\infty} f_{ii}^{(k)} \underbrace{\sum_{s=1}^{\infty} f_{ii}^{(s)}}_{f_{ii}}$$

$$= f_{ii} \sum_{k=1}^{\infty} f_{ii}^{(k)} = (f_{ii})^2$$

$$P(\text{At least 2 returns to } i \mid X_0 = i) = [f_{ii}]^2$$

Nice, huh?

So let M = # of returns to i

$$P(M \geq k \mid X_0 = i) = f_{ii}^k \qquad k \geq 1$$

Thus $$\boxed{E[M \mid X_0 = i] = \sum_{k=1}^{\infty} P(M \geq k \mid X_0 = i)}$$

$$= \sum_{k=1}^{\infty} (f_{ii})^k = \boxed{\frac{f_{ii}}{1 - f_{ii}}} \quad \text{(geometric)}$$

Notice that as $f_{ii} \to 1$ $E[M \mid X_0 = i] = +\infty$.

What does this say about recurrent states.

RECURRENT STATE $\Leftrightarrow$ $f_{ii} = 1$

$$\left(P(M \geq k \mid X_0 = i) = f_{ii}^k = 1 \right)$$

★ HW: Worth proving: if $X \geq 0$, $X \in \{0, 1, \dots\}$

$$E[X] = \sum_{k=1}^{\infty} P(X \geq k).$$

Another formula for expected # of returns:

$$E[M \mid X_0 = i] = E\left[\sum_{n=1}^{\infty} \mathbb{1}_{\{X_j = i\}} \mid X_0 = i\right]$$

$$= \sum_{n=1}^{\infty} P(X_j = i \mid X_0 = i)$$

$$= \sum_{n=1}^{\infty} P_{ii}^{n}$$

Theorem: A state i is recurrent iff

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = +\infty$$

[This theorem works even if S is countable]

If a state is transient

$$\sum_{n=1}^{\infty} P_{ii}^{n} < \infty \quad (\Rightarrow P_{ii}^{n} \to 0)$$

So if a chain has transient states and a stationary prob. π then $\pi_i = 0$ if i is transient.

Example: ∞ one dimensional walk.

$q = 1-p$

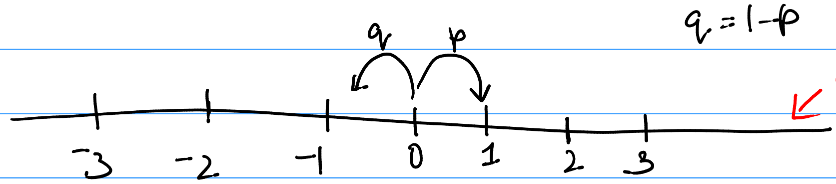


Fig 1

$S = \{0, \pm 1, \pm 2, \cdots\}$ Q: Is 0 recurrent?

$$P_{00}^{(n)} = P(X_n = 0 \mid X_0 = 0)$$

"choose $\frac{n}{2}$ steps to go right and $\frac{n}{2}$ steps to go left"

$$= P\left[\overbrace{\bigcup_{\substack{\text{all paths} \\ \gamma \text{ starting} \\ \text{and ending at } 0}} \gamma \text{ has } \tfrac{n}{2} \text{ right steps and } \tfrac{n}{2} \text{ left steps}}^{*1}\right]$$

$$= \sum_{\gamma} P(\underbrace{\gamma \text{ has } \tfrac{n}{2} \text{ right steps and } \tfrac{n}{2} \text{ left steps}}_{}) \quad - ☆1a$$

$$\text{`` } p^{n/2} q^{n/2}$$

Eg. $P(X_1 = 1, X_2 = 2, X_4 = 1, \ldots \mid X_0 = 0)$

$$= P(X_1 = 1 \mid X_0 = 0)\, P(X_2 = 2 \mid X_1 = 1) \cdots$$

$$= p^{n/2} q^{n/2}$$] using the independence of steps.

☆1a becomes

$$P_{00}^{(n)} = \binom{n}{n/2} p^{n/2} q^{n/2} \qquad ☆2$$

Use stirling: $n! \sim n^{n+1/2} e^{-n} \sqrt{2\pi}$

$$\binom{n}{n/2} = \frac{n!}{\frac{n}{2}!\,\frac{n}{2}!} \sim \frac{\cancel{n}^{n+1/2} e^{-n} \sqrt{2\pi}}{\left(\left(\frac{\cancel{n}}{2}\right)^{\frac{n}{2}+1/2} e^{-n/2} \sqrt{2\pi}\right)^2}$$

$$= \left[\frac{2^{n+1}}{\sqrt{2\pi}} \frac{1}{\sqrt{n}}\right] = C \frac{2^n}{\sqrt{n}} \qquad ☆3$$

Then ☆3 becomes

$$\approx C \frac{2^n}{\sqrt{n}} p^{n/2} q^{n/2} = \frac{C}{\sqrt{n}} (4pq)^{n/2}$$

If $(4pq) < 1$ Then

The series $\sum P_{00}^{(n)} = \sum_{n=1}^{\infty} \frac{(4pq)^{n/2}}{\sqrt{n}} < \infty$

(TRANSIENT)

and if $(4pq) > 1$ $\sum_{n=1}^{\infty} P_{00}^{(n)} = +\infty$

(RECURRENT)

In the $p = q = \frac{1}{2}$ case, $4pq = 1$

$$\sum_{n=1}^{\infty} P_{00}^{(n)} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = +\infty$$

*HW to prove these things using integral test and other series comparison.

In the SYMMETRIC CASE $\Rightarrow$ 0 is RECURRENT

Is state 5 recurrent? (Same argument works)

$i \leftrightarrow j$ and i recurrent $\Rightarrow$ j recurrent.

Lec.19

Proof of theorem 4.2

Recall that we said that

M_i = # of visits to state i

$= \sum_{n=1}^{\infty} 1_{\{X_n = i\}}$

$M_1 = 1_{\{X_0=1\}} + 1_{\{X_1=1\}} + \cdots$ (0, 1, 0)

$(0, 1, 2, 3, 2, 3, 4, \ldots)$

$M_1 = 1 \quad M_2 = 2$
$M_3 = 3$

Then $E[M_i \mid X_0 = i] = \dfrac{f_{ii}}{1 - f_{ii}}$ — ☆1

which is $< \infty$ iff $f_{ii} < 1$ $\quad f_{ii} = P(\text{visit } i \text{ at least once} \mid X_0 = i)$

i is RECURRENT if $f_{ii} = 1$.

But $f_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)}$ and $f_{ii}^{(n)}$ may be pretty hard

Thm 4.2 i is RECURRENT iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$.

Pf: $M_i = \sum_{n=1}^{\infty} 1_{\{X_n = i\}}$

☆1 $= E[M_i \mid X_0 = i] = \sum_{n=1}^{\infty} \overbrace{E[1_{\{X_n = i\}} \mid X_0 = i]}^{P(X_n = i \mid X_0 = i)}$

$= \sum_{n=1}^{\infty} P_{ii}^{(n)}$ (expectation of indicator is a probability)

☆ Good quiz/exam problem

But from previous we know $E[M_i \mid X_0 = i]$

$= \infty$ iff i is recurrent!

<u>Corollary</u> If $i \leftrightarrow j$ and i is recurrent, then so is j

communicate

Pf: $\exists$ n and m st $P_{ij}^{(n)}, P_{ji}^{(m)} > 0$.

enough to show this is ∞

Markov property

$$\sum_{\nu=0}^{\infty} P_{jj}^{(m+n+\nu)} \geq \sum_{\nu=0}^{\infty} \underbrace{P_{ji}^{(m)} P_{ii}^{(\nu)} P_{ij}^{(n)}}$$

represents the probab of all the ways of going from j to j in $m+n+\nu$ steps

represents the prob of going from j to i in m steps, then ii in ν and and then i to j in n.

$$= P_{ji}^{(n)} P_{ij}^{(m)} \underbrace{\sum_{\nu=0}^{\infty} P_{ii}^{(\nu)}}_{=+\infty} . \quad \star 1$$

So if the RHS of $\star 1$ is ∞ then so is the LHS

$\Rightarrow$ j is recurrent.

This proves that RECURRENCE is a CLASS PROPERTY (All the states in a comm. class are RECURRENT or NONE of them are recurrent)

Sec 4.4 The Basic Limit Theorem ($P^n_{ii} \xrightarrow{n\to\infty}$)

Suppose S is countable or finite.

Let $R_i = \min\{n \geq 1; X_n = i\}$ is the first return time to i

$$f_{ii}^{(n)} = P(X_n = i, X_{n-1} \neq i, \dots X_1 \neq i \mid X_0 = i)$$

$$= P(\underbrace{R_i = n}_{\text{first return time to } i} \mid X_0 = i) \text{ (pmf of returntime)}$$

Let $m_i = E[R_i \mid X_0 = i] = \sum_{n=0}^{\infty} n P(R_i = n \mid X_0 = i)$

= "mean time to return to i given you start at state i"

$$= \sum_{n=0}^{\infty} n f_{ii}^{(n)}$$

(

Thm: Given a $\underset{d=1}{\text{aperiodic}}$ irreducible recurrent MC, Then

$$\lim_{n\to\infty} P_{ii}^{(n)} = \frac{1}{\sum_{n=0}^{\infty} n f_{ii}^{(n)}} = \frac{1}{m_i} = \pi_i \geq 0$$

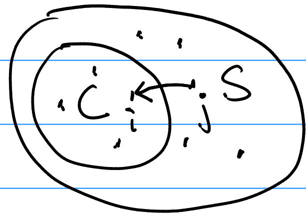
(APPLIES TO COUNTABLY ∞ # of states)

This theorem can also be applied INSIDE each recurrent class of an MC)

★ Compute this for SRW.

This argument shows that if you start in a recurrent class then you can never leave it.

Let C be a recurrent class.



Suppose $j \notin C$, $i \in C$

then 1) $P_{ji}^{(m)} = 0 \ \forall\ m$ OR

2) $P_{ij}^{(n)} = 0 \ \forall\ n$.

Suppose $P_{ij}^{(n)} > 0$ for some $n \geq 1$. $\Rightarrow P_{ji}^{(m)} = 0 \ \forall m$

[Otherwise $j \Leftrightarrow i$]

$$\{X_k \neq i \text{ for } \infty\ k,\ X_0 = i\} \supseteq \{\underset{\substack{\uparrow \\ \text{visiting } j \text{ after} \\ n \text{ steps.}}}{X_n = j},\ X_0 = i\}$$

$$P(\{X_k \neq i \text{ for } \infty\ k,\ X_0 = i\}) \geq P[\{X_n = j,\ X_0 = i\}]$$

$\Rightarrow$

$$P(\{X_k \neq i \text{ for } \infty\ k \mid X_0 = i\}) \geq P[\{X_n = j \mid X_0 = i\}] = P_{ij}^{(n)} > 0$$

$$1 - P(\{X_k \neq i \text{ for } \infty\ k \mid X_0 = i\}^c) \geq P_{ij}^{(n)}$$

$$\Rightarrow \underbrace{P(X_k = i \text{ for infinitely many } k \mid X_0 = i)} \leq 1 - P_{ij}^{(n)} < 1$$

But this is the event we require to have full probability for recurrence.

So:

<u>Prop</u>: If $i \in C$, $j \notin C$ and C recurrent then

$P_{ij}^{(n)} = 0 \quad \forall n.$

$$\begin{array}{cc} & \begin{array}{cc} 1\ 2\ \cdots\ k & \cdots\cdots \end{array} \\ \text{all in } C \left[\begin{array}{c} 1 \\ 2 \\ \vdots \\ k \end{array}\right] & \left[\begin{array}{c:c} & 0\,0\,0\,0\,0 \\ & 0\ \cdots\ 0 \\ & 0\ \cdots\ 0 \\ & \\ & \end{array}\right] \end{array}$$

You can look at each recurrent class and treat it as its own MC (since you can never leave it)

Remember I said that an absorbing state is like Hotel California?

Thats a bad analogy. Really RECURRENT CLASSES are like Hotel California. Each state in the class is like a room in the hotel.

Let $\pi_i = \lim_{n \to \infty} P_{ii}^{(n)} = \frac{1}{\mu_i}$ in an aperiodic recurrent Markov chain.

π_i could well be 0. When is it not?

POSITIVE RECURRENCE: An aperiodic recurrent class is called Positive recurrent.

if $P_{ii}^{(n)} \longrightarrow \pi_i > 0$

Or $m_i = E[R_i \mid X_0 = i] < \infty$

(The EXPECTED Return time is finite)

(NULL RECURRENT) if $P_{ii}^{(n)} \longrightarrow 0$

Thm: In a positive recurrent aperiodic class with a finite or countably many states

$\lim_{n\to\infty} P_{jj}^{(n)} = \pi_j$ where π_j determined uniquely by

$\underbrace{\pi_i > 0, \ \sum \pi_i = 1 \quad \pi_j = \sum_i \pi_i P_{ij}}$ ☆3

$\pi = (\pi_1, \pi_2, \dots)$ vector solution to these equations

STATIONARY DISTRIBUTION

1 (eigenvalue)

$\pi = \pi P$

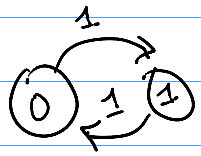
If $\lim_{n\to\infty} P_{ki}^{n} = \pi_i$ then we know π_i is a stationary

BUT even π_i is a stationary distribution, (satisfies ☆3)

we need not have

$$\boxed{\lim_{n\to\infty} P_{ii}^{(n)} = \pi_i}$$

Ex: $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad P^n = \begin{cases} I & n \text{ is even} \\ P & n \text{ is odd} \end{cases}$



$\lim_{n\to\infty} P^n$ does not exist BUT.

$$\overset{\pi}{(1/2 \quad 1/2)} \overset{P}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \overset{\lambda=1}{} \overset{\pi}{(1/2 \quad 1/2)}$$

π EXISTS.

(The problem here is periodicity)

The reason P^n_{ii} does not converge is because

$d_0 = 2 = d_1$.

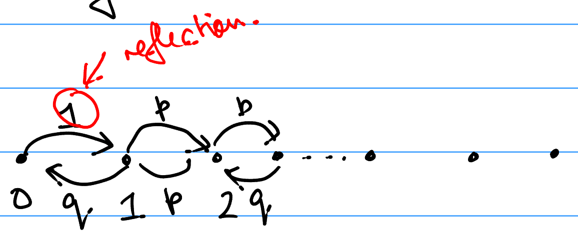
$$\lim_{n\to\infty} P_{11}^{(nd)} = 1 \qquad \lim_{n\to\infty} P_{11}^{(nd+1)} = 0$$

Thm: In a positive recurrent PERIODIC class with a finitely many states

$$\frac{1}{d} \sum P_{ij}^{(n+1)} + \cdots P_{ij}^{n+d} \xrightarrow{n\to\infty} \pi_j$$

(instead of $P^n_{ij} \to \pi_j$)

Semi-Infinite Random walk with reflecting boundary



$p + q = 1$

$$P = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & \cdots \\ 1 & q & 0 & p & 0 \; \cdots \\ 2 & 0 & q & 0 & p \; \cdots \end{array}$$

∞ matrix.

Find stationary distribution.

$\pi_0 = \pi_1 q$

$\pi_1 = \pi_0 + q\pi_2$

$\pi_2 = p\pi_1 + q\pi_3$

$\pi_3 = p\pi_2 + q\pi_4$

$\rightarrow \pi_1 = q^{-1}\pi_0$

$\pi_2 = q^{-1}\pi_1 p = pq^{-2}\pi_0$

$\pi_3 = p^2 q^{-3}\pi_0$

$\pi_k = p^{k-1}q^{-k}\pi_0$

$$\pi_0 + \pi_1 + \cdots = \pi_0\left[1 + q^{-1} + pq^{-2} + \cdots (p/q)^{k-1}q^{-1} \cdots\right] = 1$$

ONLY CONVERGES IF $p < q$.

$$\pi_0\left[1 + q^{-1}\frac{1}{1 - pq^{-1}}\right] = 1 \Rightarrow \pi_0\left[\frac{1 + q - p}{q - p}\right] = 1$$

geom.

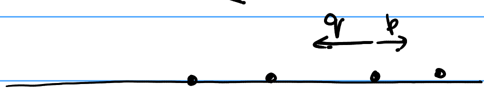
$$\pi_0 = \frac{q - p}{1 + q - 1 + q} = \frac{q - p}{2q} = \frac{1}{2}\left[1 - p/q\right]$$

$$\pi_k = q^{-1}(pq^{-1})^{k-1} \overbrace{\frac{1}{2}(1 + pq^{-1})}^{\pi_0}$$

$$\pi_k = C\left(\frac{p}{q}\right)^{k-1} = C'\left(\frac{p}{q}\right)^{k}$$

This looks a lot like a geometric. Is this a geometric? HW or QUIZ.] Think about it

$\longleftarrow$ Leftward drift towards the origin.



Success Runs Markov Chain

$$P = \begin{array}{c|c} & \begin{array}{cccccc} 0 & \;1 & \;2 & & & \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \\ \end{array} & \begin{bmatrix} p & q & 0 & 0 & \cdots & \\ p & 0 & q & 0 & \cdots & \\ p & 0 & 0 & q & \cdots & \\ & & & & & \end{bmatrix} \end{array}$$

Determine if Recurrent or NOT.

How many recurrence classes? ONLY 1.

R_0 = 1st return time to 0.

$$P(R_0 > 1 \mid X_0 = 0) = q$$

$$P(R_0 > 2 \mid X_0 = 0) = q^2$$

$$P(R_0 > k \mid X_0 = 0) = q^k$$

Ek. $P(R_0 > k \mid X_0 = 0) = 1 - \sum_{n=1}^{k} f_{00}^{(n)}$

since $f_{00}^{(n)}$ is the pmf of R_0

$$\lim_{k\to\infty} q^k = \lim_{k\to\infty} 1 - \sum_{n=1}^{k} f_{00}^{(n)}$$

$$\Rightarrow \quad 0 = 1 - \sum_{n=1}^{\infty} f_{00}^{(n)} \quad \Rightarrow \text{0 is RECURRENT}$$

Since the chain is irreducible. ALL states RECURRENT.

★ HW or Quiz: Show all states communicate.

Summary: In countable, aperiodic, recurrent classes, we've stated

$$\lim_{n\to\infty} P_{ji}^{(n)} = \frac{1}{m_i} = \pi_i \begin{cases} > 0 & \text{if positive recurrent} \\ = 0 & \text{called null recurrent} \end{cases}$$

Will prove this in the finite state case

And established 2 conditions for recurrence.

1) $\underbrace{\sum_{n=1}^{\infty} f_{00}^{(n)}}_{\text{return time is finite}} = 1$ 2) $\underbrace{\sum_{n=1}^{\infty} P_{ii}^{(n)}}_{\text{mean \# of recurrences is } \infty} = +\infty$

PERIOD

Periodic case: Here: $(d > 1)$

$$P_{ii}^{m} = 0 \quad \text{if} \quad m \neq nd$$

$$\lim_{n \to \infty} P_{ii}^{nd} = \frac{d}{m_i} = d\pi_i$$ limit exists along subsequences

mean return time to i

$m_i = E[R_i \mid X_0 = i]$

$$\Rightarrow \lim_{n \to \infty} \underbrace{\frac{1}{d} \sum_{m=0}^{d-1} P_{ii}^{(n+m)}} = \pi_i$$

mean # of visits to site i

Maybe worth skipping.

Lec 20 : (linear algebra approach to MC with finitely many states)

Recall our 2 state MC where we found

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$ has evals 1 and $1-a-b$.

$\det(\lambda I - P) = 0$

We found Q the matrix of row eigenvectors.

$$Q = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{1}{a+b} & \frac{-1}{a+b} \end{pmatrix}$$ and its inverse $\underbrace{Q^{-1}}_{\text{column eigenvectors}}$

Then $QPQ^{-1} = \Lambda$ diagonal matrix of eigenvalues.

Why did this work? We wrote

left evec of $\lambda=1$

$$Q = \begin{bmatrix} v_1 \\ \hdashline v_2 \end{bmatrix}$$ where $v_1 P = v_1$, $v_2 P = (1-a-b)v_2$

evec $\lambda = 1-a-b$

$$Q^{-1} = \left[\vec{x}_1 \,\vdots\, \vec{x}_2\right]$$ $\vec{x}_1$ and $\vec{x}_2$ are $\underbrace{\text{some column vectors}}_{\text{right column eigen vectors}}$

$$QQ^{-1} = \begin{bmatrix} v_1 \\ \hdashline v_2 \end{bmatrix} \left[\vec{x}_1 \,\vdots\, \vec{x}_2\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$= \begin{bmatrix} v_1 \cdot \vec{x}_1 & v_1 \vec{x}_2 \\ v_2 \cdot \vec{x}_1 & v_2 \cdot \vec{x}_2 \end{bmatrix}$$ (dot products of $v_i \cdot \vec{x}_j$

$\lambda=1$

$\boxed{\vec{v}_1} \perp \boxed{\vec{x}_2} \to \lambda = 1-a-b$

$\vec{v}_2 \perp \vec{x}_1$

$\Rightarrow \quad \vec{v}_1 \cdot \vec{x}_1 = 1 \qquad v_1 \cdot \vec{x}_2 = 0$

$\qquad \vec{v}_2 \cdot \vec{x}_1 = 0 \qquad v_2 \cdot \vec{x}_2 = 1$

So $\quad QP = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} P = \begin{bmatrix} v_1 \cdot P \\ v_2 \cdot P \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1 v_1 \\ \lambda_2 v_2 \end{bmatrix} \qquad \lambda_1 = 1 \quad \lambda_2 = 1-a-b$$

$$\Rightarrow \quad QPQ^{-1} = \begin{bmatrix} \lambda_1 v_1 \\ \lambda_2 v_2 \end{bmatrix} \left[\vec{x}_1 \,\vdots\, \vec{x}_2 \right] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

The upshot is that we can compute

$$P^n = Q^{-1} \Lambda^n Q \qquad \Lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$$

and so we could explicitly COMPUTE P^n.

This works more generally.

Note Q^{-1} consists of the RIGHT eigenvectors of P

So take any P with N states. We know $(1,1,1\ldots 1)$ is a RIGHT eigenvector.

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ & & \\ p_{n1} & & p_{nn} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} p_{11} + \cdots + p_{1n} \\ \vdots \\ \vdots \end{pmatrix}_{n\times 1}$$

$$= \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

So 1 is ALWAYS an eigenvalue.

$$P\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \overset{1}{\overset{\shortparallel}{(\lambda)}}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \leftarrow \text{right eigenvector.}$$

for any P.

<u>Perron-Frobenius</u>: Suppose P has ALL POSITIVE entries, then

1) $\lambda_1 = 1$ is a simple eigenvalue

2) All other evalues have $|\lambda_i| < 1$

3) $\exists$ a LEFT eigenvector corresponding to $\lambda = 1$ has all positive entries.] existence of stationary probability

<u>lemma</u>: Its enough if $\exists M$ st $n \geq M \Rightarrow P^n$ has all positive entries.

$$P_{2\times 2} = Q^{-1} \Lambda Q$$

$$P^n = Q^{-1} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} Q$$

$\Rightarrow$ I can compute P^n once I find left eigenvectors Q, Q^{-1} and the eigenvalues.

$P_{n\times n}$.

$P_{ij} > 0 \ \forall i,j$ ($\Rightarrow$ P is regular)

(Meaning it has multiplicity 1).

$\Rightarrow$ Eigenspace is 1 dimensional.

$$C(\lambda) = \det(P - \lambda I) = \pm(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}$$

$\Rightarrow$ $\lambda_1, \ldots \lambda_k$ are eigenvalues

$m_1, \ldots m_k$ are multiplicities.

$\lambda = 1$, $m_1 = 1$ (1 is a SIMPLE eigenvalue)

(If P is regular, Then $\exists k$ st $P^k_{ij} > 0$. So Perron-Frobenius applies to P^k

OK, so what? Then we can write

$P = Q^{-1} \Lambda Q$ where Λ is in JORDAN FORM

(close to diagonal)

Thus

$$\Lambda = \left[\begin{array}{c|c} 1 & 0 \cdots \\ \hline 0 & M \end{array}\right] \quad m_1 = 1$$

$$\Lambda^n = \left[\begin{array}{c|c} 1^n & 0 \\ \hline 0 & M^n \end{array}\right]$$ and since

the eigenvalues of P are all < 1 except for 1

$M^n \to 0$.

Thus $\lim_{n\to\infty} P^n = Q^{-1} \lim_{n\to\infty} \Lambda^n Q = Q^{-1} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right] Q$

How do you perform this multiplication?

$P^k = Q^{-1} \Lambda^k Q$ where.

$$\Lambda^k = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & \cdots \\ \hline 0 & \lambda_2^k & 0 & 0 \\ \hline 0 & 0 & \lambda_3^k & 0 \\ \hline 0 & 0 & 0 & \ddots \end{array}\right]$$ Jordan form.

$$\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = Q^{-1} \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1-a-b \end{array}\right] Q$$

$$P = Q^{-1} \begin{bmatrix} \boxed{\lambda_1 \ddots \lambda_1} & 0 & 0 & 0 \\ 0 & \boxed{\lambda_2 \ddots \lambda_2} & 0 & 0 \\ 0 & 0 & \boxed{\lambda_3 \ddots \lambda_3} & 0 \\ 0 & 0 & 0 & \lambda \ddots \lambda \end{bmatrix} Q$$

m_1, m_2, m_1, m

$$\begin{bmatrix} \frac{1}{2} & 0.1 \\ 0 & \frac{1}{2} \end{bmatrix} = A$$

Recall that $Q = \begin{bmatrix} \cdots \vartheta_1 \cdots \\ \vdots \\ \cdots \vartheta_n \cdots \end{bmatrix}$ left eigenvector correspons to $\lambda_1 = 1$

LEFT eigenvectors.

Will show $Q^{-1} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$ right eigenvectors.

Correspond to right eigenvectors with the same corresponding eigenvalue as the left:

$$\boxed{v_k \cdot P = \lambda_k \vartheta_k \qquad P c_k = \lambda c_k}$$

Moreover eigenvectors are unique upto multiplication constant.

Note $v_i P = \lambda_i v_i$, $P c_k = \lambda_k c_k$

Assume $i \neq k$ thus

$v_i P c_k = \lambda_i v_i \cdot c_k = \lambda_k v_i \cdot c_k$

$\lambda_i \neq \lambda_k$ (distinctness) Then $v_i \perp c_k$

Perron Frobenius $\Rightarrow$ 1 is simple and UNIQUE.

So $v_1 \perp c_k \ \forall k \neq 1$ and $\Leftrightarrow (v_1, c_k) = 0$

$v_k \perp c_1 \ \forall k \neq 1 \quad \Leftrightarrow (v_k, c_1) = 0$

$(\vartheta_k)_{1 \times n} \cdot P_{n \times n} \cdot c_j$

row matrix column

$\underbrace{\vartheta_i}_{1\times n} \ \underbrace{P}_{n\times n} \ \underbrace{c_k}_{n\times 1} = \lambda_i \vartheta_i \cdot c_k$

$= \vartheta_i (\lambda_k c_k) = \lambda_k \vartheta_i \cdot c_k$

$\Rightarrow (\lambda_i - \lambda_k)(\vartheta_i \cdot c_k) = 0$

$\Rightarrow \lambda_i = \lambda_k$ X $\Rightarrow \vartheta_i \cdot c_k = 0$

$\Rightarrow \vartheta_i$ and c_k are ORTHOGONAL $\vartheta_i \perp c_k$

Now Q is invertible and

$$Qx = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (\text{since } QQ^{-1} = I)$$

Thus x is unique. We have shown that

$$\begin{bmatrix} \vartheta_1 \\ \vdots \\ \vartheta_n \end{bmatrix} \cdot c_1 = \begin{bmatrix} \vartheta_1 \cdot c_1 \\ \vdots \\ \vartheta_n \cdot c_1 \end{bmatrix} = \begin{bmatrix} \vartheta_1 c_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Since c_1 is the eigenvector corresponding to 1

It must be $\begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix}$. Using $\vartheta_1 = (\pi_1 \cdots \pi_N)$

and $\vartheta_1 \cdot c_1 = a \sum_{i=1}^{N} \pi_i = a \Rightarrow a = 1$

So: THE FIRST COLUMN OF Q^{-1} MUST BE ALL ONES $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

So
$$Q^{-1} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\lim_{n\to\infty} \Lambda^n} Q = \begin{bmatrix} 1 & 0 & & \\ 1 & & \ddots & \\ \vdots & & & 0 \\ 1 & & & \end{bmatrix} \overbrace{\begin{bmatrix} \vartheta_1 \\ \vdots \\ \vartheta_n \end{bmatrix}}^{Q}$$

$$= \begin{bmatrix} \vartheta_1 \\ \vdots \\ \vartheta_1 \end{bmatrix} = \begin{bmatrix} \pi \\ \vdots \\ \pi \end{bmatrix}$$

π ← stationary prob. vector.

$$[1\,0\,0 \cdots 0] \begin{bmatrix} \vartheta_1 \\ \vdots \\ \vartheta_n \end{bmatrix} = 1 \cdot \vartheta_1 + 0 \cdot \vartheta_2 \cdots = \vartheta_1$$

$Ax = b$ Then if A is invertible then $x = A^{-1} b$ is UNIQUE solution to this equation.

$$QQ^{-1} = I \qquad \begin{bmatrix} \vartheta_1 \\ \vdots \\ \vartheta_n \end{bmatrix} \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}$$

$$= \begin{bmatrix} \vartheta_1 c_1 & \vartheta_1 c_2 & \cdots & \vartheta_1 \cdot c_n \\ \vartheta_2 \cdot c_1 & \cdots & & \\ \vdots & & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ & & \cdots & 0 \; 1 \end{bmatrix}$$

$\Rightarrow \vartheta_1 c_1 = 1$
$\vartheta_2 c_1 = 0$
$\vartheta_3 c_1 = 0$
$\vdots$

c_1 must of the form $a\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$\vartheta_1 \cdot c_1 = 1$ and $\vartheta_1 = (\pi_1, \cdots \pi_n)$

$\Rightarrow \vartheta_1 c_1 = a(\pi_1 + \pi_2 + \cdots + \pi_n) = 1$

$\Rightarrow a = 1$

$$\lim_{n\to\infty} P^n = Q^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} Q$$

$$\Rightarrow \begin{bmatrix} 1 & c_2 \cdots c_n \\ \vdots & \\ 1 & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ \vdots & & & \\ 0 & 0 & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

row vector · Matrix · row vector

$$[x_1 \cdots x_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = [x_1 v_1 + x_2 v_2 + \cdots x_n v_n]$$

Last fact we need:

If P is regular, P^n has all positive entries. $\Rightarrow$ PERRON Frobenius applies.

$\Rightarrow$ P^n has 1 as a simple eval. All other evalues have $|\lambda| < 1$.

Let us suppose P has k distinct eigenvalues and k corresponding eigenvectors.

Then $P v_i = \lambda_i v_i$

$\Rightarrow$ $P^n v_i = P^{n-1} P v_i = P^{n-1} \lambda_i v_i = \lambda_i^n v_i$

$\Rightarrow$ λ_i^n is the corresponding eval of P^n

by Perron Frobenius $|\lambda_i^n| < 1$ except for $\lambda_1 = 1$

A similar argument using the Jordan form shows that $\lambda_1 = 1$ is simple.

PERIODIC CASE: In this case the Perron-Frob theorem says that there are d eigenvalues such that $|\lambda_i| = 1 \quad i = 1, \ldots, d$

HW: $P = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0 & 0.4 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$

Using a computer, find $P_{21}^{7}, P_{24}^{8}, P_{21}^{20}$

Compute the long term prob of being in state 1 using the stationary probability

HW: 1.5 from lawler. Find all communication classes.

Last few topics for long term behavior

1) Martingales and MC → Lawler Ch 2 and Ch 5

2) A condition for transience. → Partially in KP Ch 4

2a) Apply this theory to the 1d random walk

3) Random walk in Higher d. → Lawler Ch 2 as well

Martingale Suppose we have an MC on a countable # of states.

Let

Harmonic for the MC

$f: S \to \mathbb{R}$ st $f(x) = \sum_{y \in S} p(x,y) f(y)$

Then $f(X_n)$ is a Martingale.

$\mathcal{F}_n = (X_1, \dots, X_n)$

$\mathbb{E}[f(X_n) \mid \mathcal{F}_{n-1}] = \mathbb{E}[f(X_n) \mid X_{n-1}]$

$\mathbb{E}[f(X_n) \mid X_{n-1} = x] = \sum_{y \in S} \overbrace{p(x,y)}^{\mathbb{P}(X_n = y \mid X_{n-1} = x)} f(y)$

$= f(x)$

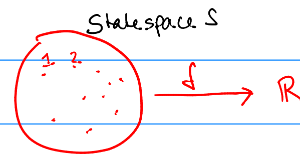
$\Rightarrow \mathbb{E}[f(X_n) \mid X_{n-1}] = f(X_{n-1})$

ASSUME ONLY ONE CLASS. (Chain is irreducible)

Fix a state $z \in S$. Define

$\alpha(x) = P(X_n = z \text{ for some } n \geq 0 \mid X_0 = x)$

$= P(\text{ever reach } z \text{ given you start at } x)$



$p(x,y) = \mathbb{P}(X_{n+1} = y \mid X_n = x)$

$= P_{xy}$

$E[f(X_n) \mid X_1, \dots X_{n-1}] = f(X_{n-1})$

$= \underbrace{E[f(X_n) \mid X_{n-1}]}_{g}$

$g(i) = \mathbb{E}[f(X_n) \mid X_{n-1} = i]$

Need to show: $E[f(X_n) \mid X_1 = a_1, \dots, X_{n-1} = a_{n-1}]$

$= f(a_{n-1})$

$= \sum P(X_n = y \mid X_{n-1} = x) f(y)$

$p(x,y) = \mathbb{P}(X_n = y \mid X_{n-1} = x)$

$= P_{xy}$

$\alpha(z) = 1.$

If $x \neq z$

$\alpha(x) = P(X_n = z \text{ for some } \overline{n \geq 1} \mid X_0 = x)$

First step analysis

$$= \sum_{y \in S} p(x,y) P(\underbrace{X_n = z \text{ for some } n \geq 1 \mid X_1 = y}_{=1 \text{ if } y = z})$$

$$= p(x,z) + \sum_{y \neq z} p(x,y)\alpha(y)$$

$\alpha(z) = 1$

$$= p(x,z)\alpha(z) + \sum_{y \neq z} p(x,y)\alpha(y)$$

$$= \sum_{y \in S} p(x,y)\alpha(y)$$

α is almost harmonic

$\alpha(x) = \sum p(x,y)\alpha(y) \quad x \neq z$

$\alpha(z) = 1 \neq \sum_{y \in S} p(z,y)\alpha(y)$] $\alpha(z) \neq \sum_{y \in S} p(z,y)\alpha(y)$

BUT that is OK for us.

$\alpha(x) = \sum p(x,y)\alpha(y)$ Then $\alpha(X_n)$ is a martingale.

Properties of α

1) $0 \leq \alpha \leq 1$] since its a probability

2) $\alpha(x) = \sum p(x,y)\alpha(y) \quad x \neq z$

3) $\alpha(z) = 1$

Let $T = \min\{n \geq 0 \mid X_n = z\}$] First time at which you hit state z

Then $\alpha(x) = P(T < \infty \mid X_0 = x) = P(X_n = z \text{ for some } n \geq 0 \mid X_0 = x)$

Let $T_n = \min(n, T)$ Then $\alpha(X_{T_n})$ is a martingale, since:

$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \min(n,T) = T$ if T is finite

$$E[\alpha(X_{T\wedge n}) \mid \mathcal{F}_{n-1}] \overset{\text{Markov}}{=} E[\alpha(X_{T\wedge n}) \mid X_{T\wedge n-1} = x]$$

$E[\alpha(X_{T_n}) \mid \mathcal{F}_{n-1}] = \alpha(X_{T_{n-1}})$

Martingale property.

When $x \neq z$, $\Rightarrow T \geq n-1$ $= E[\alpha(X_n) \mid X_{n-1} = x]$

(if $T < n-1$ $X_{T\wedge n-1} = X_T = z \neq x$)

$$= \underset{\text{yes}}{\sum} \alpha(y)\, p(x,y) \underset{\alpha \text{ is Harmonic}}{=} \alpha(x)$$

$$= \alpha(X_{T\wedge n-1})$$

When $x = z$ then

$$E[\alpha(X_{T\wedge n}) \mid X_{T\wedge n-1} = z] = \alpha(z) = \alpha(X_{T\wedge n-1})$$

($\Rightarrow T \leq n-1$) $X_{T\wedge n} = X_T = z$

$\longrightarrow$ Proves that $\alpha(X_{T\wedge n})$ is a martingale.

Then $E[\alpha(X_{T\wedge n})] = E[\alpha(X_{T\wedge 0})] = E[\alpha(X_{T\wedge n})]$ $\quad T\wedge 0 = 0$ since $T \geq 0$.

let $n \to \infty$

$= \alpha(X_0)$ ★1 (if $X_0 = x$)

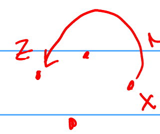
uses bounded convergence theorem. (α is bounded)

Take $\lim_{n\to\infty} \mathbb{E}\left[\alpha(X_{T\wedge n})\right] = \mathbb{E}\left[\lim_{n\to\infty} \alpha(X_{T\wedge n})\right]$

$\lim_{n\to\infty} \alpha(X_{T\wedge n}) = \alpha(X_T)$ assuming $P(T<\infty \mid X_0 = x) = 1$

$= \mathbb{E}\left[\alpha(X_T)\right] = \alpha(z) = 1$ (Markov Chain at stopping time must be at z)

Assuming $P(T<\infty \mid X_0 = x) = 1$
(we will prove this on an HW) ☆.



reach z in finite time or $T<\infty$.

So if an irreducible chain is recurrent
$\alpha(x) = 1 \quad \forall\, x \in S$

$\alpha(x) = P(\, z \text{ is reached in finite time} \mid X_0 = x)$
$= 1$

$1 = \alpha(z) = \mathbb{E}\left[\alpha(X_T)\right] = \alpha(X_0) = \alpha(x) \quad x \neq z$

$\Rightarrow \alpha = 1$

So: if a chain is recurrent, there is no

$\alpha : S \to \mathbb{R}$ st for any fixed z
1) $\alpha(x) = \sum p(x,y)\alpha(y) \quad x \neq z$
2) $0 \le \alpha \le 1$
3) α is not identically 1

There is not way to find an α satisfying these equations if our chain is recurrent (using Martingales)

If you can find such a fn + an extra condition then the chain is transient to

Theorem I "sketched" in the finite case $\left[\; \underbrace{\pi = \pi P}_{\text{Recurrence}} \quad \underbrace{P\alpha = \alpha}_{\text{Transience}} \right]$

except for $x \neq z$.

Condition for Transience:

Turns out: if an irreducible chain is transient, $\exists\ \alpha$ st (UNIQUE)

1) $\alpha(x) = \sum p(x,y)\alpha(y)$ $\quad P\alpha = \alpha$

2) $0 \leq \alpha \leq 1$

3) $\inf_{x \in S} \{\alpha(x)\} = 0$

ANALOGS

$\pi = \pi P$ (S countable as well).

$$[\ \cdots\ ] = [\ \cdots\ ]\,[\ \cdots\ ]$$

$\pi(i) = \sum \pi(j) P(i,j)$ ★1 $\quad \sum \pi(i) = 1$

If a solution exists st $\pi(i) > 0\ \forall i$, then chain positive recurrent.

$P\alpha = \alpha$ $\quad [\ \cdots\ ]\,[\ \ ] = [\ \ ]$

$\sum P_{ij}\,\alpha(j) = \alpha(i)$ ★2

$\sum \pi(x) = 1$

$\pi(x) > 0 \ \forall x \in S.$

To summarize :

1) If an irreducible chain is positive recurrent then $\exists\ \pi_j$ st

$$\sum \pi_j = 1 \ , \ \pi_j > 0 \quad \pi(x) = \sum_{y \in S} \pi(y) P(y,x)$$

$$\lim_{n\to\infty} P_{ij}^{n} \longrightarrow d\,\pi_j$$

If you can find such a π then we KNOW ITS +ve recurrent.

If NO such solution exists

$\Downarrow$

2) A chain is transient is $\exists\ \alpha$ st (UNIQUE)

1) $\alpha(x) = \sum p(x,y)\alpha(y)$

2) $0 \leq \alpha \leq 1$

3) $\inf_{x} \{\alpha(x)\} = 0$

If you can find such an α then the chain is TRANSIENT.

$\Downarrow$

Else NULL recurrent.

Another way of directly detecting NULL recurrence is to show that

$P(T_x \mid X_0 = x) = 1$ BUT $E[T_x \mid X_0 = x] = +\infty$

where T_x = 1st return time to x.

Prob of returning to x at least once given $X_0 = x$ $= 1$

BUT average time to return is $+\infty$.

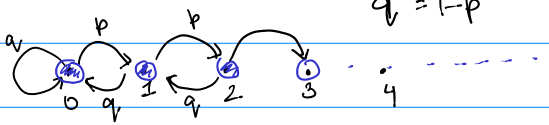
(Avg time to return if +ve recurrent is
$E[T_x \mid X_0 = x] = \frac{1}{\pi(x)} < \infty$)

lec 22 More on the Random Walk.

We now wish to determine whether or not the SRW in 1d is NULL Recurrent or transient.

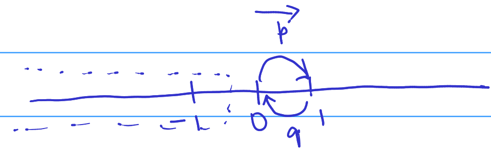
We do this by studying the "partially reflecting SRW"

$q = 1-p$



Claim: The recurrence and transience of 0 in the SRW on $\mathbb{Z}$ can be studied with this model.

→ Partially reflecting at 0.



↳ Is 0 recurrent in SRW on $\mathbb{Z}$?

$\sum P_{00}^{(n)} \approx \sum \frac{c}{\sqrt{n}} = +\infty \Rightarrow$ SRW was recurrent $p = q$.

1) Let's try to find α. ☆1

$$\alpha(x) = \sum p(x,y)\alpha(y) = q\,\alpha(x-1) + p\,\alpha(x+1)$$

We have solved such equations in the past.

$\vec{P}\vec{\alpha} = \alpha \qquad \sum P_{ij}\,\alpha_j = \alpha_i$

$\pi(x) = p\,\pi(x-1) + q\,\pi(x+1)$

$\pi(x) = C_1\left(\frac{p}{q}\right)^x + C_2$ | general.

We have seen a version of this in lec 19

Turns out that the solution is

(*3) $\alpha(x) = c_1 + c_2\left(\frac{q}{p}\right)^x \qquad p \neq \frac{1}{2} \ (p \neq q)$

$\alpha(x) = c_1 + c_2 x \qquad p = \frac{1}{2} \ (p = q)$

All solutions can be written as linear combs of $(q/q)^x$ and 1

Theory of linear difference equations.

Can we find α st:

1) $0 \leq \alpha(x) \leq 1 \quad \forall x$

2) $\inf_x \alpha(x) = 0$

3) $\alpha(0) = 1 \quad (z = 0)$

$\alpha(0) = 1 \Rightarrow$ (*3) $c_1 + c_2 = 1$

$q < p$

$\alpha(x) = (q/p)^x$ which satisfies all 3

$q < p$.

$p \neq 1/2 \qquad \alpha(0) = 1 \Rightarrow c_1 + c_2 = 1]$

2) $\Rightarrow \frac{q}{p} < 1 \Rightarrow p > \frac{1}{2}$ (or $c_2 = 0$ which does not work)

$\lim_{x \to \infty} c_1 + c_2 \left(\frac{q}{p}\right)^x = c_1 \Rightarrow c_1 = 0, \ c_2 = 1$

$\left(\frac{q}{p}\right)^x \to 0 \ (q < p)$

THUS $p > \frac{1}{2} \Rightarrow$ transient.

In $p = q$ there must be no solution

$p = q \qquad \alpha(x) = c_1 + c_2 x \qquad \alpha(0) = 1 \Rightarrow c_1 = 1$

$\Rightarrow \alpha = 1$

$0 \leq \alpha \leq 1 \Rightarrow c_2 = 0$, $\boxed{\text{No solution.}}$

Can it be positive RECURRENT?

Solved in the fully reflecting case.

Now we have to solve

$$\pi = \pi P$$

$$\begin{array}{c} \pi(x) \\ \left[\cdots\cdots \right] \end{array} = \begin{array}{c} \pi \\ \left[\quad\quad \right] \end{array} \begin{array}{c} P(y,x) \\ \left[\begin{array}{c} \vdots \\ \vdots \end{array} \right] \end{array}$$

$$\pi(x) = \sum_{y \in S} \pi(y) P(y,x)$$

Here $\pi(x) = p\,\pi(x-1) + q\,\pi(x+1)$

$$\left[\alpha(x) = q\,\alpha(x-1) + p\,\alpha(x+1) \right]$$

we have done this lec 19 explicitly.

So this time

$$\pi(x) = \begin{cases} c_1 + c_2 \left(\frac{p}{q}\right)^x & p \neq q \\ c_1 + c_2 x & p = q \end{cases}$$

$$\underbrace{\sum \pi(x) = 1 \Rightarrow (p/q)^x \text{ to decay} \Rightarrow p<q}$$

$c_1 = 0 \quad c_2 = \dfrac{1}{\sum_{x=0}^{\infty} (p/q)^x}, \quad \pi(x) = c_2 (p/q)^x > 0.$

For it to work this time, we must have $p<q$ since otherwise we cannot satisfy $\sum_x \pi(x) = 1$

If $p=q$ avg. time to return to 0 is $+\infty$

So we have discovered that

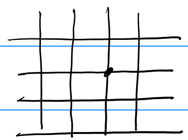
$p<q$ POSITIVE RECURRENT ✓ (found π)

$p>q$ TRANSIENT ✓ (found α)

$p=q$ NULL RECURRENT (only possibility left in the $p=q$ case.)

We can use this case to make similar statements about the unrestricted 1D Random walk.

Random Walk in general dimensions



We again write

$X_n = \xi_1 + \cdots + \xi_n \qquad X_0 = 0$

where ξ_i are iid and take

values in $\{ \underbrace{(1,0\cdots)}_{\text{length } d}, (-1,0,\cdots 0), (0,1,\cdots\cdots) \}$

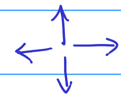
There are d components to each step ξ_i

We want to determine whether or not 0 is recurrent.

$\boxed{\text{★ What is the periodicity of the } d \text{ dimensional random walk.}}$

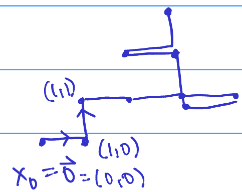
$d=2$, $d=3$ and so on.

(Symmetric random walk)



In $d=2$, $2d=4$ possibilities for steps.

$\underbrace{\sum P_{00}^{(n)} \overset{?}{=} +\infty}_{\text{Recurrent}}$ and see when $\underbrace{\sum P_{00}^{(n)} < \infty}_{\text{Transient.}}$



Again recall that

$$\sum P_{00}^{(n)} = +\infty \iff 0 \text{ is recurrent.}$$

★ HW or Quiz: in a recurrent class, can you determine $P(X_n = i \text{ for some } n \geq 0 \mid X_0 = i)$

As in 1D, we need the # of steps to be even: $2n$

Of these approximately $\frac{2n}{d}$ are in component 1, $\frac{2n}{d}$ are in component 2, ...

Say $d=7$ and $n=100$ ($2n=200$), then d may not divide $2n$ exactly.

Let $A_1, A_2, \dots, A_d$ be the # of steps taken in each comp.

So we only know that $A_1 + A_2 + \dots + A_d = 2n$

and LLN says $A_1 \approx A_2 \approx \dots \approx A_d \approx \frac{2n}{d}$

Since there are "an equal # of even and odd #s" the probability that A_i is even is approximately $\left(\frac{1}{2}\right)$. One can also show that these A_i are approximately independent.

Thus $P(A_1, A_2, \dots A_d \text{ are all even}) \approx \left(\frac{1}{2}\right)^{d-1}$

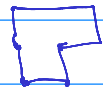
In each of these components, you want the steps to sum to 0, so like in 1d this has probability

Can we 1d asym for component?

$$\sum P_{00}^{(2n)} = P(X_{2n}=0 \mid X_0=0) = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \approx \frac{1}{\sqrt{\pi n}} = \frac{c}{\sqrt{n}}$$

# of paths with equal up and down steps

Prob of any such path $p^n q^n$



n steps in component 1 to be even

n steps in component 2 to be even.

$n=2$, $2n=4$

(Law of large numbers ensures that # of steps in each component is APPROX. n)

$$P(A_1, A_2, \dots, A_d \text{ are all even}, \textstyle\sum A_i = 2n) = P(A_1, A_2, \dots, A_{d-1} \text{ are all even}, \textstyle\sum A_i = 2n)$$

$A_i \approx \frac{2n}{d}$

$$\approx \frac{1}{\sqrt{\pi n}} \quad (\text{for } 2n \text{ steps})$$

Since we're taking $\frac{2n}{d}$ steps this prob is approximately

$$\approx \frac{1}{\sqrt{\pi n/d}} = \frac{c}{\sqrt{n/d}}$$

Since the components are independent.

$$\mathbb{P}(\text{all components sum to } 0) \approx \left(\frac{1}{\sqrt{\pi n/d}}\right)^{d} \quad \leftarrow d \text{ components}$$

(given approx. $\frac{2n}{d}$ steps, all even)

$$= \left(\frac{d}{\pi n}\right)^{d/2}$$

Thus $$P_{00}^{(2n)} \approx \boxed{\frac{1}{2^{d-1}}} \left(\frac{d}{\pi n}\right)^{d/2}$$

$$\sum_{n=1}^{\infty} \frac{c}{n^{d/2}}$$

$$\int_1^{\infty} \frac{dx}{x^{d/2}} = \left.\frac{x^{-d/2+1}}{-d/2+1}\right|_1^{\infty}$$

$$= 2\left(\frac{d/2}{\pi n}\right)^{d/2} = \frac{c}{n^{d/2}}$$

$$= \left.\frac{c}{x^{d/2-1}}\right|_1^{\infty} \quad d/2 \neq 1 \quad d/2 > 1$$

$$\int_1^{\infty} \frac{dx}{x} = \log x \Big|_1^{\infty} = +\infty$$

$$\sum_{n=1}^{\infty} P_{00}^{(2n)} = \begin{cases} +\infty & d = 1,2 \quad \left(\frac{c}{\sqrt{n}}, \frac{c}{n}\right) \quad \text{RECURRENT} \\ <\infty & d = 3,4,\dots \quad \left(\frac{c}{n^{3/2}}, \dots\right) \quad \text{TRANSIENT} \end{cases}$$

$$\int_1^{\infty} \frac{1}{x^{\alpha}}\,dx = \begin{cases} \log x \Big|_1^{\infty} & \alpha = 1 \\ \left.\frac{(1-\alpha)}{x^{\alpha-1}}\right|_1^{\infty} & (= \infty \text{ unless } \alpha > 1) \end{cases}$$

SUMMARY

1) A drunk man or woman will always find their way home $(d = 2)$

2) A drunk bird will never find its way home $(d = 3)$